Tutorial 2

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Outline

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"Rationalizing Denominator"

Question: How to rationalize the denominator of $\frac{1}{\sqrt{2}+1}$?

Hint: Use $\sqrt{2}-1$.

Answer: $\frac{\sqrt{2}-1}{(\sqrt{2}+1)(\sqrt{2}-1)} = \sqrt{2}-1$.

Question: How to "rationalize" the denominator of $\frac{1}{1+i}$?

Hint: Use $1-i$ (Conjugate).

Answer: $\frac{1-i}{(1+i)(1-i)} = \frac{1-i}{2}$.

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Examples

Example

$z_1 = 8 + 3i$ and $z_2 = 9 - 2i$, find $\frac{z_1}{z_2}$.

Solution $\frac{z_1}{z_2} = \frac{8+3i}{9-2i} = \frac{(8+3i)(9+2i)}{(9-2i)(9+2i)} = \frac{66}{85} + \frac{43}{85}i$.

Exercise

$z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, find $\frac{z_1}{z_2}$.

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Identities of Conjugate

The complex conjugate $\bar{z}$ of a complex number $z = x + iy$ is defined by $\bar{z} = x - iy$.

Identities 1

- $Re(z) = x = \frac{1}{2}(z + \bar{z})$,
- $Im(z) = y = \frac{1}{2i}(z - \bar{z})$,
- $z\bar{z} = x^2 + y^2$.

Identities 2

- $\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2$,
- $\overline{(z_1 - z_2)} = \bar{z}_1 - \bar{z}_2$,
- $\overline{(z_1 z_2)} = \bar{z}_1 \bar{z}_2$,
- $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$.

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Examples

Example

Let $z_1 = 4+3i$ and $z_2 = 2+5i$. Verify that $\overline{(z_1 z_2)} = \bar{z}_1 \bar{z}_2$.

Solution

$$\overline{(z_1 z_2)} = \overline{(4+3i)(2+5i)} = \overline{(-7+26i)} = -7-26i$$
$$\bar{z}_1 \bar{z}_2 = (4-3i)(2-5i) = -7-26i$$

Exercise

Verify the other identities for $z_1 = 4+3i$ and $z_2 = 2+5i$.

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1. Complex Numbers
   - Division
   - Complex Conjugate Numbers
   - Polar Form of Complex Numbers

2. Derivative
   - Monotonicity and the Sign of the Derivative

Definitions

- Polar coordinates r, θ defined by $x = r\cos\theta$, $y = r\sin\theta$.
- $z = x + iy$ takes the so-called **polar form** $z = r(\cos\theta + i\sin\theta)$
  - The absolute value or modulus of z , r, is denoted by $|z|$. $|z| = r = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$.
  - The argument of z, θ, is denoted by $\arg z$. $\theta = \arg z = \arctan \frac{y}{x}$.

Examples

Example

Let $z = 1 + i$, find the polar form of z.

Solution $z = \sqrt{2}(\cos\frac{1}{4}\pi + \sin\frac{1}{4}\pi)$. Hence, $r = |z| = \sqrt{2}$ and $\arg z = \frac{1}{4}\pi \pm 2n\pi$ $(n = 0, 1, \ldots)$.

Exercise

Let $z = 3 + 3\sqrt{3}\mathrm{i}$, find the polar form of z.

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The Geometrical Meaning of Multiplication and Division

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

Multiplication

$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$(Why??)

- $|z_1 z_2| = |z_1||z_2|$
- $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

Division

$z_1/z_2 = r_1/r_2[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$(Why??)

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Powers and Roots

Integer powers of z

$z^n = r^n(\cos n\theta + i\sin n\theta)$ (why??)

*De Moivre's formula: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

nth root of z

$\sqrt[n]{z} = \sqrt[n]{r}(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n})$, where $k = 0, 1, \ldots, n-1$. (Why??)

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Puzzle

What is i^i?

Hint: What is $e^{i\frac{\pi}{2}}$?

Answer

$e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$

$\Longrightarrow \ln i = i\frac{\pi}{2} \Longrightarrow i\ln i = -\frac{\pi}{2}$

Since $i^i = e^{i\ln i}$, therefore $i^i = e^{-\frac{\pi}{2}}$.

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Definitions and Rules

- Definition
  - A function f(x) is increasing on an interval I iff for any $x_1, x_2 \in I$ $x_1 \leq x_2$ implies $f(x_1) \leq f(x_2)$.
  - A function f(x) is decreasing on an interval I iff for any $x_1, x_2 \in I$ $x_1 \leq x_2$ implies $f(x_1) \geq f(x_2)$.
- Rules (Why?? Hint: definition of derivative)
  - $f'(x) \geq 0$ for all x in the interior of I exactly if $f(x)$ is increasing on all of I;
  - $f'(x) \leq 0$ for all x in the interior of I exactly if $f(x)$ is decreasing on all of I;
  - $f'(x) = 0$ for all x in the interior of I exactly if $f(x)$ is constant on all of I.

Example

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Sketch the function $f(x) = 2x^3 - 3x^2 - 12x + 1$

Solution $f'(x) = 6x^2 - 6x - 12 = 6(x-2)(x-1)$.

So we have , $\begin{cases} f'(x) > 0 & \text{if } x < -1 \\ f'(x) < 0 & \text{if } -1 < x < 2 \\ f'(x) > 0 & \text{if } x > 2 \\ f'(x) = 0 & \text{iff } x = -1 \text{ or } x = 2 \end{cases}$

So $f(x)$ is increasing on the intervals $(-\infty, -1]$ and $[2, \infty)$, and $f(x)$ is decreasing on the interval $[-1, 2]$.

Verify your sketch by the maxima code:

wxplot2d([2*x^3-3*x^2-12*x+1], [x,-5,5],[y,-20,10])$

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Exercises

Exercise1

Find the intervals on which $f(x) = x + \sin x$ is increasing or decreasing.

Exercise2

Without calculating the numerical results, can you tell me which number is larger, e^{π} or π^{e}? (Hint: 1. Monotonicity of a function or 2. Taylor's series of e^{x})