1 Double Integral

A double integral is something of the form

$$\iint_R f(x,\ y)dxdy$$

where R is called the region of integration and is a region in the $(x,\ y)$ plane.

Geometrical Meaning: The double integral gives us the volume under the surface $z = f(x,\ y)$, just as a single integral gives the area under a curve.

1.1 Calculation of double integrals

- Work out the limits of integration if they are not already known (most difficult part)
- Work out the inner integral
- Work out the outer integral (using our knowledge of the methods for single integrals)

1.2 The limits are given (limits are constants)

Example 1. Calculate

$$\int_{y=1}^{2}\int_{x=0}^{3}(1+8xy)dxdy$$

Solution.

$$\begin{aligned}\text{integral} &= \int_{y=1}^{2}\left(\underbrace{\int_{x=0}^{3}(1+8xy)dx}_{\text{work out treating } y \text{as constant}}\right)dy\\ &= \int_{y=1}^{2}\left[x+4x^2y\right]_{x=0}^{3}dy\\ &= \int_{y=1}^{2}(3+36y)dy\\ &= \left[3y+18y^2\right]_{y=1}^{2}\\ &= 57\end{aligned}$$

Exercise 1. Calculate

$$\int_0^{\pi/2}\int_0^1 y\sin x dydx$$

(Answer: $\frac{1}{2}$)

1.3 The limits are given (limits are not constants)

Example 2. Calculate

$$\int_0^2 \int_{x^2}^x y^2 x dy dx$$

Solution.

$$\begin{aligned}
\text{integral} &= \int_0^2 \int_{x^2}^x y^2 x dy dx \\
&= \int_0^2 \left[\frac{y^3 x}{3}\right]_{y=x^2}^{y=x} dx \\
&= \int_0^2 \left(\frac{x^4}{3} - \frac{x^7}{3}\right) dx \\
&= \left[\frac{x^5}{15} - \frac{x^8}{24}\right]_0^2 \\
&= -\frac{128}{15}
\end{aligned}$$

Exercise 2. Calculate

$$\int_{\pi/2}^{\pi} \int_0^{x^2} \frac{1}{x} \cos \frac{y}{x} dy dx$$

(Answer: 1)

Exercise 3. Calculate

$$\int_1^4 \int_0^{\sqrt{y}} e^{x/\sqrt{y}} dx dy$$

(Answer: $\frac{14}{3}(e-1)$)

1.4 The limits are not given

When calculating double integrals, it is very common not to be told the limits of integration but simply told that the integral is to be taken over a certain specified region R in the $(x,\ y)$ plane. In this case you need to work out the limits of integration for yourself. Great care has to be taken in carrying out this task.

The integration can in principle be done in two ways:

- integrating first with respect to x and then with respect to y,
- first with respect to y and then with respect to x.

The limits of integration in the two approaches will in general be quite different, but both approaches must yield the same answer. Sometimes one way round is considerably harder than the other, and in some integrals one way works fine while the other leads to an integral that cannot be evaluated using the simple methods you have been taught. There are no simple rules for deciding which order to do the integration in.

Example 3. Calculate

$$\int\int_D (4x+2)dA, \ (dA \text{ means } dxdy \text{ or } dydx)$$

where D is the region enclosed by the curves $y = x^2$ and $y = 2x$.

Solution 1. A good diagram is essential!

We do the integration first with respect to x and then with respect to y. We shall need to know where the two curves $y = x^2$ and $y = 2x$ intersect. They intersect when $x^2 = 2x$, i.e. when $x = 0,\ 2$. So they intersect at the points $(0,0)$ and $(2,4)$. For a typical y, the horizontal line will enter D at $x = y/2$ and leave at $x = \sqrt{y}$. Then we need to let y go from 0 to 4 so that the horizontal line sweeps the entire region. Thus

$$\begin{aligned}
\int\int_D (4x+2)dA &= \int_0^4 \int_{x=y/2}^{x=\sqrt{y}} (4x+2)dxdy \\
&= \int_0^4 \left[2x^2+2x\right]_{x=y/2}^{x=\sqrt{y}} dy \\
&= \int_0^4 \left((2y+2\sqrt{y}) - \left(\frac{y^2}{2}+y\right)\right) dy \\
&= \int_0^4 \left(y + 2y^{1/2} - \frac{y^2}{2}\right) dy \\
&= \left[\frac{y^2}{2} + \frac{2y^{3/2}}{3/2} - \frac{y^3}{6}\right]_0^4 = 8
\end{aligned}$$

Solution2. Integrate first with respect to y and then x, i.e. draw a vertical line across D at a typical x value. Such a line enters D at $y = x^2$ and leaves at $y = 2x$. The integral becomes

$$\begin{aligned}
\int\int_D (4x+2)dA &= \int_0^2 \int_{x^2}^{2x} (4x+2)dydx \\
&= \int_0^2 \left[4xy+2y\right]_{y=x^2}^{y=2x} dx \\
&= \int_0^2 ((8x^2+4x) - (4x^3+2x^2))dx \\
&= \int_0^2 (6x^2 - 4x^3 + 4x)dx \\
&= \left[2x^3 - x^4 + 2x^2\right]_0^2 = 8
\end{aligned}$$

Exercise 4. Calculate

$$\int\int_D (3-x-y)dA, \ (dA \text{ means } dxdy \text{ or } dydx)$$

where D is the triangle in the (x,y) plane bounded by the x-axis and the lines $y = x$ and $x = 1$.
(Answer: 1)

Exercise 5*. Calculate

$$\int\int_D (xy - y^3)dA, \ (dA \text{ means } dxdy \text{ or } dydx)$$

where D is the region consisting of the square $\{(x,y) : -1 \leq x \leq 0, 0 \leq y \leq 1\}$ together with the triangle $\{(x,y) : x \leq y \leq 1, 0 \leq x \leq 1\}$.

(Answer: $-\frac{23}{40}$)

Exercise 6*. Calculate

$$\int\int_D \frac{\sin x}{x} dA, \ (dA \text{ means } dxdy \text{ or } dydx)$$

where D is the triangle $\{(x,y) : 0 \leq y \leq x, 0 \leq x \leq \pi\}$.

(Answer: 2)

2 Finding the maximum/minimum of a function

Consider a continuously differentiable function $f(x,y)$.

2.1 Unconstrainted optimization

The domain of $f(x,y)$ is R^2. (all points in (x,y) plane)

If f has a maximum/minimum, then the maximum/minimum must occur when $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$. Let (x^*, y^*) be the point such that $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$.

Checking for the maximum or minimum:

Hessian: is the square matrix of second-order partial derivatives of a function $f(x,y)$.

$H(f) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$.

- if the determinant is positive at (x^*, y^*), then f attains a local minimum at (x^*, y^*).
- if the determinant is negative at (x^*, y^*), then f attains a local maximum at (x^*, y^*).

Example 4. Find the maximum/minimum of $f(x,y) = x^2 + 2y^2 - 2xy + 4x$.

Solution. $\begin{cases} \frac{\partial f}{\partial x} = 2x - 2y + 4 = 0 \\ \frac{\partial f}{\partial y} = 4y - 2x = 0 \end{cases} \Rightarrow \begin{cases} x = & -4 \\ y = & -2 \end{cases}$ and $f(-4,-2) = 8$

Because the determinant of

$H(f) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}$ is positive, then $f(-4,-2) = -8$ is a local minimum.

2.2 Constrainted optimization

The domian of $f(x,y)$ is D.

First, we should get the point (x^*, y^*) such that $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$.

Then, if $(x^*, y^*) \in D$, check the value of f at both (x^*, y^*) and the boundary.

If $(x^*, y^*) \notin D$, check the value of f at the boundary.

Example 5. Find the maximum/minimum of $f(x,y) = x^2 - 5xy + y$, D is bounded by $x = 0$, $y = 2$ and $y = x^2$.

Solution. $\begin{cases} \frac{\partial f}{\partial x} = 2x - 5y = 0 \\ \frac{\partial f}{\partial y} = -5x + 1 = 0 \end{cases} \Rightarrow \begin{cases} x = & \frac{1}{5} \\ y = & \frac{2}{25} \end{cases}$ $(\frac{1}{5}, \frac{2}{25}) \in D$ and $f(\frac{1}{5}, \frac{2}{25}) = \frac{1}{25}$.

Check the boundary:

1. $x = 0$, $0 \le y \le 2$, $f(0,y) = y \in [0,2]$
2. $y = 2$, $0 \le x \le \sqrt{2}$, $f(x,2) = x^2 - 10x + 2 \in [4 - 10\sqrt{2}, 2]$
3. $y = x^2$, $0 \le x \le \sqrt{2}$, $f(x,y) = f(x,x^2) = 2x^2 - 5x^3$, which increases when $x \in [0, \frac{4}{15}]$ and decreases when $x \in [\frac{4}{15}, \sqrt{2}]$. So $f(x,x^2) \in [4 - 10\sqrt{2}, \frac{32}{675}]$.

Thus, the minimum is $4 - 10\sqrt{2}$ and occurs at $(\sqrt{2}, 2)$.

The maximum is 2 and occurs at $(0,2)$.